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ON A THEORY FOR LARGE ELASTIC DEFORMATION OF SHELLS OF REVOLUTION INCLUDING TORSION AND THICK-SHELL EFFECTS

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Abstract—This paper presents a set of equations governing the behavior of elastic shells of revolution undergoing large axisymmetric deformations. The approximate theory includes torsion, transverse shear deformation, and transverse normal stress and strain but neglects body forces. Within the limits of large membrane and moderately large bending and transverse shear strains, a two-dimensional strain–energy density function is developed for a general incompressible, hyperelastic shell material.

INTRODUCTION

During the last quarter of a century, advances in shell theory have brought valuable new insights into a complex subject. Two-dimensional formulations have yielded particularly elegant derivations of the governing equations, especially as regards the non-linear theory (Reissner, 1974). Within the confines of axisymmetric deformation of shells of revolution with torsion, this paper examines the "exact" two-dimensional non-linear equations in the light of making simplifications consistent with a descent from three-dimensional elasticity theory. Torsion can be important, for example, in fiber-wound pressure vessels with off-axis orthotropy. Reissner (1966, 1969a, 1970) undertook similar work within the realm of the linear theory.

A key advance in the formulation of a general non-linear shell theory was the approach proposed by Simmonds and Danielson (1972) and later expounded upon by Reissner (1974) and Libai and Simmonds (1983). In particular, Simmonds and Danielson split the general deformation into a rigid-body rotation, which carries an orthogonal triad of unit vectors at a point $\bar{P}$ into a second orthogonal triad at its deformed image P, and a distortion, which is defined by the difference between the deformed base vectors and the triad at P. Pietraszkiewicz (1980) and Schmidt (1985) discuss similar decompositions for geometrically non-linear shell theories. Besides enabling a straightforward derivation of a set of intrinsic shell equations, this approach can be used as a basis for a non-linear theory of elasticity (Reissner, 1975). The current work uses this type of formulation as a convenient means to establish the connection between the two- and three-dimensional equations.

Much recent research has turned to the development of an appropriate form for the strain–energy density function to be used in shell problems involving large elastic strains. In a descent from three dimensions, Simmonds (1985) obtained a first approximation for general rubber-like shells, with later specialization and simplification of the analysis for torsionless, axisymmetric deformation of shells of revolution (Simmonds, 1986). The basic kinematic assumption of the theory allows transverse normal strains but neglects transverse shear strains, which were added later (Taber, 1987). To complement the approximate field equations, this paper modifies and further extends the earlier work on strain–energy functions to include torsion in a quasi-linear theory, i.e. membrane strains can be large, but bending and transverse shear strains can be only moderately large.

The descent from three dimensions brings at least one new question to light. A shell theory based strictly on two-dimensional arguments does not explicitly account for the stress components σ_{3i} ($i = 1, 2, 3$), which are not necessarily equal to σ_{i3}, where "3" is the transverse direction. In the linear theory, Reissner (1970) showed that these components

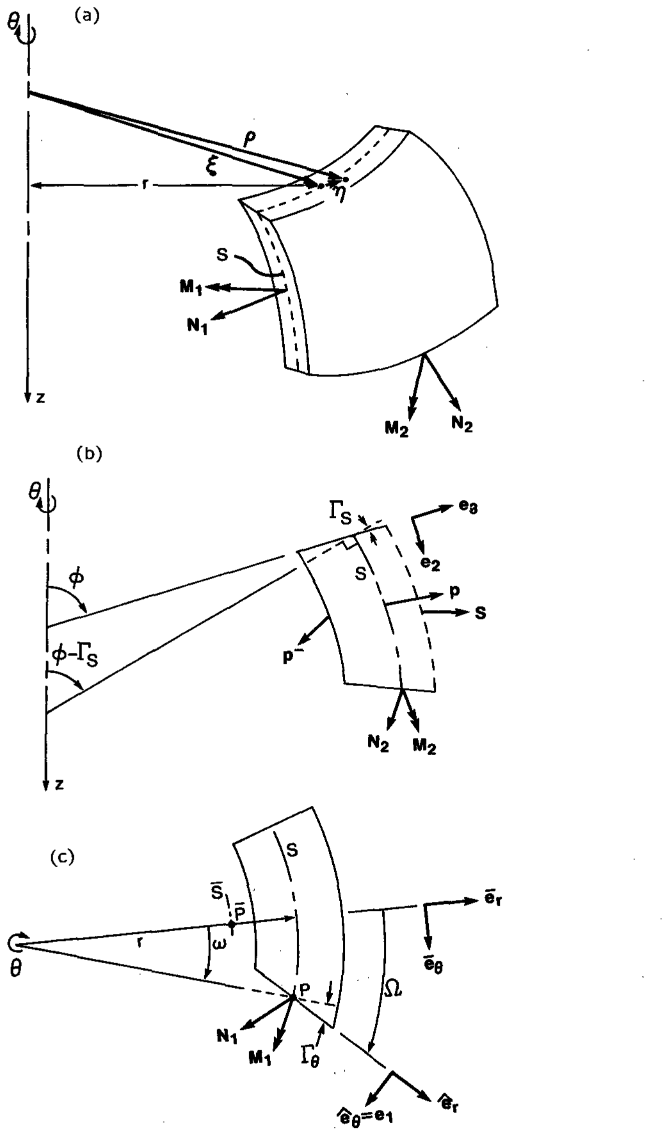

Fig. 1. Deformed shell element: (a) three-dimensional schematic; (b) side view; (c) top view.

can be eliminated with the aid of the constitutive relations. But, if these equations are nonlinear and material dependent, then the difficulty, if there is one, apparently is not resolved so simply. This paper, therefore, also examines the question of whether there is any significant inconsistency in neglecting these stresses in a two-dimensional shell theory.

FUNDAMENTAL EQUATIONS

Geometry

Deformation. Consider a shell with an axis of symmetry z (Fig. 1) defined relative to a global cylindrical coordinate system with unit vectors $(\bar{\mathbf{e}}_r, \bar{\mathbf{e}}_\theta, \bar{\mathbf{e}}_z)$. A point $\bar{P}$ on the undeformed reference (middle) surface $\bar{S}$ has the coordinates $(\bar{r}, \bar{\theta}, \bar{z})$ where bars indicate the undeformed state. Associated with $\bar{P}$ is a local orthogonal triad of unit vectors $\bar{\mathbf{e}}_i$, which are parallel to the coordinate lines $(\bar{\theta}, \bar{s}, \bar{n}) \equiv (\bar{x}_1, \bar{x}_2, \bar{x}_3)$, with $\bar{s}$ being the meridional and $\bar{n}$ the transverse normal direction.† During deformation, $\bar{P}$ moves to P on the deformed reference surface S, which is not necessarily the geometric midsurface. A second local

† In this paper, unless indicated otherwise, the usual summation convention on repeated subscripts applies where Latin subscripts range over 1, 2, 3 and Greek subscripts over 1, 2. Also, it will be understood that $A_{,1} \equiv \partial A/\bar{r}\,\partial\theta$.

orthogonal triad $\mathbf{e}_i$, obtained from $\bar{\mathbf{e}}_i$ through a rigid-body rotation, is associated with P. As suggested by Simmonds and Danielson (1972), the $\mathbf{e}_i$ will represent the basic reference frame.

Lines normal to $\bar{S}$ and S enclose the angles $\bar{\phi}$ and $\phi-\Gamma_s$, respectively, with the axis of symmetry. The meridional *face* of a shell element, which is *not* assumed to remain straight, rotates through the additional average shear angle Γ_s so that it forms an average angle ϕ with the z-axis (Fig. 1(b)). Similarly, due to torsion, the circumferential face rotates through an average angle Ω about the z-axis (Fig. 1(c)), with $\omega = \Omega-\Gamma_\theta$ being the circumferential angle between P and $\bar{P}$. The rotated frame $\mathbf{e}_i$ corresponds to the deformed positions of these element faces. In terms of the defined coordinate bases, the position vectors to an arbitrary point in the shell, before and after deformation (Fig. 1(a)), are

$$\bar{\boldsymbol{\rho}} = \bar{\boldsymbol{\xi}}+\bar{n}\bar{\mathbf{e}}_3$$
$$\boldsymbol{\rho} = \boldsymbol{\xi}+\boldsymbol{\eta}, \quad \boldsymbol{\eta} = \eta_i\mathbf{e}_i \tag{1}$$

where

$$\bar{\boldsymbol{\xi}} = \bar{r}\bar{\mathbf{e}}_r+\bar{z}\bar{\mathbf{e}}_z$$
$$\boldsymbol{\xi} = r(\bar{\mathbf{e}}_r \cos\omega+\bar{\mathbf{e}}_\theta \sin\omega)+z\bar{\mathbf{e}}_z \tag{2}$$

represent the positions of the reference surface.

The frame $\mathbf{e}_i$ is obtained by rotating $\bar{\mathbf{e}}_i$ through Ω about $\bar{\mathbf{e}}_z$ into an intermediate system $\hat{\mathbf{e}}_i$, and then rotating the $\hat{\mathbf{e}}_i$ through $-(\phi-\bar{\phi})$ about $\hat{\mathbf{e}}_1$. A simpler, equivalent procedure is, with $\bar{P}$ taken at $\bar{\theta}=0$ (without loss of generality), to rotate the global frame $(\bar{\mathbf{e}}_r, \bar{\mathbf{e}}_\theta, \bar{\mathbf{e}}_z)$ through Ω about $\bar{\mathbf{e}}_z$ to get $(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z)$ and then through $\pi/2-\phi$ about $\hat{\mathbf{e}}_\theta$ to arrive at $\mathbf{e}_i$. This latter sequence of rigid-body rotations is given by

$$(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z) = \mathbf{Q}_1\cdot(\bar{\mathbf{e}}_r, \bar{\mathbf{e}}_\theta, \bar{\mathbf{e}}_z)$$
$$(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) = \mathbf{Q}_2\cdot(\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z) \tag{3}$$

where the rotator is (Libai and Simmonds, 1983)

$$\mathbf{Q}_\alpha = (2a_\alpha^2-1)\mathbf{I}+2\boldsymbol{\beta}_\alpha\boldsymbol{\beta}_\alpha+2a_\alpha\boldsymbol{\beta}_\alpha\times \quad (\alpha \textit{ not summed}) \tag{4a}$$

in which

$$a_\alpha = \cos\beta_\alpha/2, \quad \boldsymbol{\beta}_\alpha = \mathbf{E}_\alpha \sin\beta_\alpha/2 \tag{4b}$$

are Euler parameters, with β_α being the angle of rotation and $\mathbf{E}_\alpha$ the unit vector along the axis of rotation. For the current problem

$$\beta_1 = \Omega, \quad \mathbf{E}_1 = \bar{\mathbf{e}}_z$$
$$\beta_2 = \pi/2-\phi, \quad \mathbf{E}_2 = \hat{\mathbf{e}}_\theta \tag{5}$$

give

$$\hat{\mathbf{e}}_r = \cos\Omega\bar{\mathbf{e}}_r+\sin\Omega\bar{\mathbf{e}}_\theta, \quad \hat{\mathbf{e}}_\theta = -\sin\Omega\bar{\mathbf{e}}_r+\cos\Omega\bar{\mathbf{e}}_\theta, \quad \hat{\mathbf{e}}_z = \bar{\mathbf{e}}_z \tag{6a}$$
$$\mathbf{e}_1 = \hat{\mathbf{e}}_\theta, \quad \mathbf{e}_2 = \cos\phi\hat{\mathbf{e}}_r+\sin\phi\hat{\mathbf{e}}_z, \quad \mathbf{e}_3 = \sin\phi\hat{\mathbf{e}}_r-\cos\phi\hat{\mathbf{e}}_z. \tag{6b}$$

Bending strains. The strains due to bending enter by way of the curvature vectors of the deformed reference surface

$$\mathbf{k}_\alpha = k_{\alpha\beta}\mathbf{e}_3\times\mathbf{e}_\beta+c_\alpha\mathbf{e}_3, \quad \mathbf{k}_3 \equiv 0 \tag{7}$$

with the corresponding undeformed vectors given by placing bars over all quantities. Substituting eqns (6) and (7) into the Gauss–Weingarten relations

$$\mathbf{e}_{j,i} = \mathbf{k}_i \times \mathbf{e}_j \tag{8}$$

provides the geometry-dependent expressions

$$\bar{k}_{11} = \frac{\sin \bar{\phi}}{\bar{r}}, \quad \bar{k}_{22} = \bar{\phi}', \quad \bar{k}_{12} = \bar{k}_{21} = 0, \quad \bar{c}_1 = -\frac{\cos \bar{\phi}}{\bar{r}}, \quad \bar{c}_2 = 0 \tag{9a}$$

$$k_{11} = \frac{\sin \phi}{\bar{r}}, \quad k_{22} = \phi', \quad k_{12} = 0, \quad k_{21} = \Omega' \sin \phi, \quad c_1 = -\frac{\cos \phi}{\bar{r}}, \quad c_2 = -\Omega' \cos \phi \tag{9b}$$

where commas and primes denote differentiation with respect to $\bar{x}_i$ and $\bar{s}$, respectively. Through the principle of virtual work, Reissner (1974) showed that the appropriate curvature change measures are†

$$\kappa_{\alpha\beta} = k_{\alpha\beta} - \bar{k}_{\alpha\beta}, \quad \psi_\alpha = c_\alpha - \bar{c}_\alpha. \tag{10}$$

Extensional strains. In their approach, Simmonds and Danielson (1972) defined the state of strain as the difference between the deformed (non-orthogonal) base vectors and the orthogonal triad $\mathbf{e}_i$. In three dimensions, the strain (base) vectors are

$$\mathbf{G}_i = \boldsymbol{\rho}_{,i} = (\delta_{ij} + E_{ij})\mathbf{e}_j \tag{11a}$$

while in two dimensions

$$\boldsymbol{\xi}_{,\alpha} = (\delta_{\alpha\beta} + e_{\alpha\beta})\mathbf{e}_\beta + \gamma_\alpha \mathbf{e}_3 \tag{11b}$$

in which $\boldsymbol{\rho}$ and $\boldsymbol{\xi}$ are defined in eqns (1) and (2). In eqns (11), E_{ij} and $e_{\alpha\beta}$ represent strain tensors, and γ_α are transverse shear strains. Following Reissner (1970), one also calls the E_{ij} pseudostrains; physical strains $\hat{E}_{ij}$ are defined by

$$\delta_{ij} + \hat{E}_{ij} = \mu_i^{-1}(\delta_{ij} + E_{ij}) \quad (i \textit{ not summed}) \tag{12}$$

where $\mu_i = |\bar{\boldsymbol{\rho}}_{,i}|$, or

$$\mu_\alpha = 1 + \bar{n}\bar{k}_{\alpha\alpha}, \quad \mu_3 = 1 \quad (\alpha \textit{ not summed}). \tag{13}$$

Specific forms for the strain components in terms of the geometry of Fig. 1 will be detailed in a later section. The average shear angle Γ_s, which is contained implicitly in eqns (2), can be found from the additional geometric relation $r' \sin(\phi - \Gamma_s) = z' \cos(\phi - \Gamma_s)$ or

$$\tan \Gamma_s = \frac{r' \sin \phi - z' \cos \phi}{r' \cos \phi + z' \sin \phi}. \tag{14}$$

Stress

The pseudostress vectors

$$\boldsymbol{\sigma}_i = \sigma_{ij}\mathbf{e}_j \tag{15}$$

describe the three-dimensional state of stress in a shell, and the physical stresses per unit *undeformed* area are

† The expressions for $\kappa_{\alpha\beta}$ used in Taber (1987) contain reference surface stretch ratios in the $\bar{k}_{\alpha\beta}$ terms. For a thin shell, these terms, which were derived on physical grounds, contribute higher-order effects in the equilibrium equations as derived through the principle of virtual work.

$$\hat{\boldsymbol{\sigma}}_1 = \mu_2^{-1}\boldsymbol{\sigma}_1, \quad \hat{\boldsymbol{\sigma}}_2 = \mu_1^{-1}\boldsymbol{\sigma}_2, \quad \hat{\boldsymbol{\sigma}}_3 = (\mu_1\mu_2)^{-1}\boldsymbol{\sigma}_3. \tag{16}$$

With t being the initial shell thickness, stress and moment resultant vectors per unit *undeformed* length of the reference surface

$$\mathbf{N}_\alpha = \int_{-t/2}^{t/2} \boldsymbol{\sigma}_\alpha \, d\bar{n}, \quad \mathbf{S} = \int_{-t/2}^{t/2} \boldsymbol{\sigma}_3 \, d\bar{n}, \quad \mathbf{M}_\alpha = \int_{-t/2}^{t/2} \boldsymbol{\eta} \times \boldsymbol{\sigma}_\alpha \, d\bar{n} \tag{17}$$

have components defined by

$$\mathbf{N}_\alpha = N_{\alpha\beta}\mathbf{e}_\beta + Q_\alpha \mathbf{e}_3, \quad \mathbf{S} = S_i\mathbf{e}_i + T\mathbf{e}_3, \quad \mathbf{M}_\alpha = M_{\alpha\beta}\mathbf{e}_3 \times \mathbf{e}_\beta + P_\alpha \mathbf{e}_3 \tag{18}$$

where T/t represents the average transverse normal stress due to surface loads directly, in addition to S_3/t due to deformation.

Equilibrium

In three dimensions, force and moment equilibrium on a deformed shell element yield the vector equations

$$(\bar{r}\boldsymbol{\sigma}_i)_{,i} = 0, \quad \boldsymbol{\rho}_{,i} \times (\bar{r}\boldsymbol{\sigma}_i) = 0 \tag{19}$$

in which body forces and the possibility of couple stresses, which were included by Reissner (1969a, 1970), have been ignored. The two-dimensional equilibrium equations are

$$(\bar{r}\mathbf{N}_\alpha)_{,\alpha} + \bar{r}\mathbf{p} = 0$$
$$(\bar{r}\mathbf{M}_\alpha)_{,\alpha} + \boldsymbol{\xi}_{,\alpha} \times (\bar{r}\mathbf{N}_\alpha) + \bar{r}\mathbf{q} = 0 \tag{20}$$

where $\mathbf{p} = p_i\mathbf{e}_i$ and $\mathbf{q} = q_i\mathbf{e}_i$ are the respective thickness-averaged traction and moment applied per unit undeformed reference surface area. The principle of virtual work gives (Libai and Simmonds, 1983)

$$\mathbf{p} = [\mathbf{p}]_-^+, \quad \mathbf{q} = [\boldsymbol{\eta} \times \mathbf{p}]_-^+, \quad \mathbf{p}^\pm \equiv (\mu_1\mu_2\hat{\mathbf{p}})^\pm \tag{21}$$

where $\hat{\mathbf{p}}$ is the physical surface traction and $+$ and $-$ denote the surfaces $\bar{n} = t/2$ and $-t/2$, respectively.

Constitutive equations

In terms of the strain and stress components defined by eqns (11a) and (15), the constitutive relations for a hyperelastic material are (Reissner, 1975)

$$\sigma_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{22}$$

in terms of the pseudostrain–energy function $W(E_{ij})$. The strain energy per unit undeformed volume is

$$\hat{W}(\hat{E}_{ij}) = (\mu_1\mu_2)^{-1} W(E_{ij}). \tag{23}$$

To this point, the development has been exact. The principle of virtual work for a strictly two-dimensional theory provides the exact two-dimensional counterpart of eqn (22) (Libai and Simmonds, 1983). The next section, however, discusses these constitutive relations within the context of an approximate descent from three dimensions.

AN APPROXIMATE SHELL THEORY

A two-dimensional representation of a three-dimensional solid necessarily contains approximations. Further approximations often must be made to render the resulting equations amenable to analysis. In the non-linear theory of shells, it is quite a challenge to formulate equations that are relatively simple yet represent the true shell behavior adequately. This section attempts to find such a set of two-dimensional relations which, for the geometry under consideration, is consistent with three-dimensional elasticity theory.

The ensuing analysis is based on expansions in terms of the small parameter

$$\varepsilon = h/L \tag{24a}$$

where $h = t/2$ and L is the smallest deformation wavelength. Following Simmonds (1986), one chooses non-dimensional quantities in the form

$$\begin{gathered}
(r^*, z^*, s^*) = L^{-1}(r, z, s), \quad \eta^*_\alpha = \eta_\alpha/h\varepsilon, \quad (n^*, \eta^*_3) = h^{-1}(n, \eta_3) \\
(k^*_{\alpha\beta}, c^*_\alpha) = L(k_{\alpha\beta}, c_\alpha), \quad (\gamma^*_\alpha, g^*_\alpha) = (\gamma_\alpha, g_\alpha)/\varepsilon \\
\sigma^*_{\alpha\beta} = \sigma_{\alpha\beta}/C, \quad (\sigma^*_{i3}, \sigma^*_{3i}) = (\sigma_{i3}, \sigma_{3i})/C\varepsilon, \quad p^*_i = p_i/C\varepsilon, \quad q^*_i = q_i/Ch\varepsilon^2 \\
N^*_{\alpha\beta} = N_{\alpha\beta}/Ch, \quad (Q^*_\alpha, S^*_i, T^*) = (Q_\alpha, S_i, T)/Ch\varepsilon \\
M^*_{\alpha\beta} = M_{\alpha\beta}/Ch^2, \quad P^*_\alpha = P_\alpha/Ch^2\varepsilon, \quad W^* = W/C, \quad w^* = w/Ch \\
(\ )' \equiv \partial(\ )/\partial\bar{s}^*, \quad (\ )^{\cdot} \equiv \partial(\ )/\partial\bar{n}^*, \quad \int \equiv \int_{-1}^{1}
\end{gathered} \tag{24b}$$

in which C is a material constant with units of a Young's modulus. This scaling assumes that transverse effects are small compared to membrane-type effects.

Geometry

From here until eqn (55), all equations are expressed in dimensionless form, but with the asterisks removed. First of all, the curvature components retain their forms given by eqns (9). Next, substitution of eqns (1), (2), and (7)–(9) into eqns (11) provides, for axisymmetry, the pseudostrain components

$$\begin{gathered}
E_1 \equiv E_{11} = e_{11} + \varepsilon\eta_3 k_{11} - \varepsilon^2\eta_2 c_1, \quad E_2 \equiv E_{12} = e_{12} + \varepsilon\eta_3 k_{12} + \varepsilon^2\eta_1 c_1 \\
E_3 \equiv E_{13} = \varepsilon\gamma_1 - \varepsilon^2(\eta_2 k_{12} + \eta_1 k_{11}), \quad E_4 \equiv E_{21} = e_{21} + \varepsilon\eta_3 k_{21} - \varepsilon^2(\eta_2 c_2 - \eta_1') \\
E_5 \equiv E_{22} = e_{22} + \varepsilon\eta_3 k_{22} + \varepsilon^2(\eta_1 c_2 + \eta_2'), \quad E_6 \equiv E_{23} = \varepsilon(\gamma_2 + \eta_3') - \varepsilon^2(\eta_2 k_{22} + \eta_1 k_{21}) \\
E_7 \equiv E_{31} = \varepsilon\dot{\eta}_1, \quad E_8 \equiv E_{32} = \varepsilon\dot{\eta}_2, \quad E_9 \equiv E_{33} = \dot{\eta}_3 - 1
\end{gathered} \tag{25}$$

where reference-surface strains are

$$\begin{gathered}
e_{11} = r\cos\Gamma_\theta/\bar{r} - 1, \quad e_{22} = (r'\cos\Gamma_\theta + r\omega'\sin\Gamma_\theta)\cos\phi + z'\sin\phi - 1 \\
e_{12} = r\cos\phi\sin\Gamma_\theta/\bar{r}, \quad e_{21} = r\omega'\cos\Gamma_\theta - r'\sin\Gamma_\theta \\
\gamma_1 = r\sin\phi\sin\Gamma_\theta/\bar{r}, \quad \gamma_2 = (r'\cos\Gamma_\theta + r\omega'\sin\Gamma_\theta)\sin\phi - z'\cos\phi.
\end{gathered} \tag{26}$$

The strain field given by eqns (25) depends on the reference-surface strains and curvatures and on $\boldsymbol{\eta}$, which describes motion relative to this surface. Matters are simplified considerably if the position of S is defined by the average

$$\boldsymbol{\xi} = \frac{1}{2}\int \boldsymbol{\rho}\, d\bar{n} \tag{27a}$$

which Libai and Simmonds (1983) introduced to obtain the dynamic equations of motion for a shell by integrating the three-dimensional relations. Substitution of eqns $(1)_2$ into eqn (27a) yields the condition

$$\int \boldsymbol{\eta}\, \mathrm{d}\bar{n} = 0. \tag{27b}$$

Consider first the components η_1 and η_2. As eqns (25) show, E_{31} and E_{32} depend only on these variables. If the shear strains are small, then, for a polar orthotropic material, the corresponding shear stresses would be proportional to them. At this point, the theory is restricted to moderately large transverse shear strains, i.e. $(E_{13}^2, E_{31}^2, E_{23}^2, E_{32}^2) \ll 1$, and it is assumed that this proportionality holds. Then, the boundary conditions

$$\bar{n} = \pm 1: \quad \sigma_{3i} = p_i^{\pm} \tag{28}$$

for $i = 1, 2$ are satisfied by $E_{31} = E_{32} = 0$ at $\bar{n} = \pm 1$ if $p_1 = p_2 = 0$.

A need to ignore the influence of the tangential components of the surface tractions on the corresponding shear strains arises due to the large-strain coupling between stretching and shear deformation (see eqn (55)). This feature precludes a simple relation between a non-zero shear stress and shear strain at the shell surfaces. On the other hand, a variational formulation can include p_1 and p_2 (Naghdi, 1957). Since three-dimensional numerical solutions, which can show the importance of these tractions, are beyond the scope of this paper, one *neglects the effects of* p_1 *and* p_2. Then, the simplified boundary conditions imply the first approximation

$$\eta_\alpha = (3g_\alpha \bar{n}/2)(1-\bar{n}^2/3) \tag{29}$$

and eqns (25) give

$$E_{3\alpha} = 2\varepsilon g_\alpha f(\bar{n}) \tag{30}$$

where

$$f(\bar{n}) = 3(1-\bar{n}^2)/4. \tag{31}$$

The g_α represent average shear strains, i.e.

$$g_\alpha = \frac{1}{2}\int E_{3\alpha}\, \mathrm{d}\bar{n}. \tag{32}$$

Now, as mentioned earlier, the g_α do not enter a strictly two-dimensional formulation. As in Timoshenko beam theory, the shear strains γ_α are often assumed constant across the thickness. On the other hand, in the linear theory, the transverse shear stresses are parabolic over the thickness with a shear correction factor entering the constitutive relations. Here, since the $E_{3\alpha}$ fields are taken parabolic (eqn (30)), the replacements

$$\gamma_\alpha \rightarrow 2\gamma_\alpha f(\bar{n}) \tag{33}$$

in eqns (25) render the $O(\varepsilon)$ terms of the $E_{\alpha 3}$ essentially in the same form, consistent with the approximate shear stress distribution (Reissner, 1970). Substitution of expression (33) into eqns (25) and noting eqn (27b) show that

$$\gamma_\alpha = \frac{1}{2}\int E_{\alpha 3}\, \mathrm{d}\bar{n} \tag{34}$$

represent average transverse shear strains. Expression (33) will be used only for the three-dimensional strain field of eqns (25).

For an incompressible material, the incompressibility condition

$$(G/\bar{G})^{1/2} = (\mathbf{G}_1 \times \mathbf{G}_2)\cdot\mathbf{G}_3/\mu_1\mu_2\mu_3 = 1 \tag{35a}$$

provides the components η_3, or, with eqns (11a) and (12), this equation becomes

$$(\delta_{1i}+\hat{E}_{1i})(\delta_{2j}+\hat{E}_{2j})(\delta_{3k}+\hat{E}_{3k})v_{ijk} = 1 \tag{35b}$$

where v_{ijk} is the three-dimensional permutation symbol. If the expansion

$$\eta_3 \equiv n = n_0+\varepsilon n_1+\cdots \tag{36}$$

and eqns (12) and (25) are inserted into eqn (35b), collection of like powers of ε and again noting eqn (27b) yield

$$n_0 = \lambda_3\bar{n} \tag{37}$$

where

$$\lambda_3 = (\lambda_1\lambda_2 - e_{12}e_{21})^{-1} \tag{38a}$$

with

$$\lambda_i = 1+e_{ii} \quad (i \text{ not summed}) \tag{38b}$$

being stretch ratios. Remarkably, due to condition (27b), this approximate theory requires only n_0. If the material is compressible, a condition such as plane stress determines η_3 (Simmonds, 1987).

Stress

Substitution of eqns $(1)_3$, (15), and (18) into eqns (17) provides the stress and moment resultant components

$$(N_{\alpha\beta}, Q_\alpha, S_\alpha) = \int (\sigma_{\alpha\beta}, \sigma_{\alpha 3}, \sigma_{3\alpha})\, \mathrm{d}\bar{n}, \quad S_3+T = \int \sigma_{33}\, \mathrm{d}\bar{n}$$

$$M_{\alpha\beta} = \int (\eta_3\sigma_{\alpha\beta} - \varepsilon^2\eta_\beta\sigma_{\alpha 3})\, \mathrm{d}\bar{n}, \quad P_\alpha = \int v_{\gamma\beta}\eta_\gamma\sigma_{\alpha\beta}\, \mathrm{d}\bar{n} \tag{39}$$

where $v_{\alpha\beta}$ is the two-dimensional permutation symbol, i.e. $v_{12} = -v_{21} = 1$, $v_{11} = v_{22} = 0$. Corresponding to the shear strains of eqns (30) and (33) are the assumed shear stress distributions

$$\sigma_{\alpha 3} = Q_\alpha f(\bar{n}), \quad \sigma_{3\alpha} = S_\alpha f(\bar{n}) \tag{40a}$$

and one takes the transverse normal stress in the form

$$\sigma_{33} = S_3 f(\bar{n}) + \tfrac{1}{2}p_3^+(1+\tfrac{3}{2}\bar{n}-\tfrac{1}{2}\bar{n}^3) + \tfrac{1}{2}p_3^-(1-\tfrac{3}{2}\bar{n}+\tfrac{1}{2}\bar{n}^3) \tag{40b}$$

where f is given by eqn (31). If

$$T = p_3^+ + p_3^- \tag{40c}$$

then eqns (40a) and (40b), which agree with the assumed stress distributions of Naghdi (1957) for the linear theory, satisfy eqns (39) for Q_α and S_i and boundary conditions (28) for $p_1 = p_2 = 0$.

Equilibrium

Combining eqns (8), (11), (15), (24), (36), and (38) with eqns (19) provides the scalar form of the three-dimensional equilibrium equations

$$\begin{aligned} f_1 &= (\bar{r}\sigma_{21})'/\bar{r} + \dot{\sigma}_{31} - \sigma_{22}c_2 - \sigma_{12}c_1 + \varepsilon(\sigma_{13}k_{11} + \sigma_{23}k_{21}) = 0 \\ f_2 &= (\bar{r}\sigma_{22})'/\bar{r} + \dot{\sigma}_{32} + \sigma_{11}c_1 + \sigma_{21}c_2 + \varepsilon(\sigma_{23}k_{22} + \sigma_{13}k_{12}) = 0 \\ f_3 &= \varepsilon(\bar{r}\sigma_{23})'/\bar{r} + \dot{\sigma}_{33} - \sigma_{11}k_{11} - \sigma_{22}k_{22} - \sigma_{12}k_{12} - \sigma_{21}k_{21} = 0 \end{aligned} \tag{41a}$$

and, to $O(\varepsilon)$ with ε divided out

$$\begin{aligned} m_1 &= \lambda_3\sigma_{32} - \lambda_2\sigma_{23} - e_{12}\sigma_{13} + e_{13}\sigma_{12} + (\gamma_2 + n_0')\sigma_{22} = 0 \\ m_2 &= \lambda_3\sigma_{31} - \lambda_1\sigma_{13} - e_{21}\sigma_{23} + e_{13}\sigma_{11} + (\gamma_2 + n_0')\sigma_{21} = 0 \\ m_3 &= \lambda_2\sigma_{21} - \lambda_1\sigma_{12} + e_{12}\sigma_{11} - e_{21}\sigma_{22} = 0. \end{aligned} \tag{41b}$$

In two dimensions, the shell equilibrium equations, given by substituting eqns (8), (11b), (18), and (24) into eqns (20), are

$$\begin{aligned} &(\bar{r}N_{21})'/\bar{r} - c_2N_{22} - c_1N_{12} + \varepsilon(k_{11}Q_1 + k_{21}Q_2) + p_1 = 0 \\ &(\bar{r}N_{22})'/\bar{r} + c_1N_{11} + c_2N_{21} + \varepsilon(k_{22}Q_2 + k_{12}Q_1) + p_2 = 0 \\ &\varepsilon(\bar{r}Q_2)'/\bar{r} - k_{11}N_{11} - k_{22}N_{22} - k_{12}N_{12} - k_{21}N_{21} + p_3 = 0 \end{aligned} \tag{42a}$$

$$\begin{aligned} &\varepsilon[(\bar{r}M_{21})'/\bar{r} - c_1M_{12} - c_2M_{22} - \lambda_1Q_1 - e_{21}Q_2 + \gamma_1N_{11} + \gamma_2N_{21}] + \varepsilon^2(k_{12}P_1 + k_{22}P_2 + q_1) = 0 \\ &\varepsilon[(\bar{r}M_{22})'/\bar{r} + c_1M_{11} + c_2M_{21} - \lambda_2Q_2 - e_{12}Q_1 + \gamma_1N_{12} + \gamma_2N_{22}] - \varepsilon^2(k_{11}P_1 + k_{21}P_2 - q_2) = 0 \\ &\varepsilon^2(\bar{r}P_2)'/\bar{r} + \varepsilon(k_{11}M_{12} + k_{21}M_{22} - k_{12}M_{11} - k_{22}M_{21}) + \lambda_1N_{12} - \lambda_2N_{21} \\ &\qquad - e_{12}N_{11} + e_{21}N_{22} + \varepsilon^2q_3 = 0 \end{aligned} \tag{42b}$$

which agree with those of Reissner (1974).

Equations (42a) also can be obtained, with the aid of eqns (21), (28), and (39), by integrating eqns (41a) over the undeformed shell thickness. On the other hand, integrating eqns $(41b)_1$ and $(41b)_2$ gives

$$\begin{aligned} S_1 &= \lambda_3^{-1}(\lambda_1Q_1 + e_{21}Q_2 - \gamma_1N_{11} - \gamma_2N_{21}) \\ S_2 &= \lambda_3^{-1}(\lambda_2Q_2 + e_{12}Q_1 - \gamma_1N_{12} - \gamma_2N_{22}) \end{aligned} \tag{43a}$$

while eqn $(41b)_3$ reproduces the first-order terms of eqn $(42b)_3$. Next, consider

$$\int f_i n \, d\bar{n} = 0$$

with the f_i given by eqns (41a). Taking $i = 1$ and 2 and noting eqns (42b) simply give eqns (43a) again. But $i = 3$ provides, to the first order

$$S_3 = -\lambda_3^{-1}(k_{11}M_{11} + k_{22}M_{22} + k_{12}M_{12} + k_{21}M_{21}) \tag{43b}$$

in which eqns (36), (37), (39), and (40) have been used. If the deformed curvatures $k_{\alpha\beta}$ are replaced by the undeformed curvatures $\bar{\kappa}_{\alpha\beta}$ and $\lambda_3 \to 1$, then, for $p_1 = p_2 = 0$, eqn (43b) reduces to the approximation for transverse normal stress derived by Naghdi (1957).

Constitutive relations

The two-dimensional counterparts of eqns (22) now will be derived. In a descent from three dimensions, the strain–energy density per unit undeformed reference surface area is

$$w = \int W \, d\bar{n}. \tag{44}$$

Differentiation of this relation with respect to each $e_{\alpha\beta}$, γ_α, g_α, $k_{\alpha\beta}$, c_α, using the chain rule, substituting eqns (10), (22), (24), (25), (30), (31), (33), and (40a), and integrating yields the first-order approximations†

$$N_{\alpha\beta} = \frac{\partial w}{\partial e_{\alpha\beta}}, \quad Q_\alpha = \varepsilon^{-2} b \frac{\partial w}{\partial \gamma_\alpha}, \quad S_\alpha = \varepsilon^{-2} b \frac{\partial w}{\partial g_\alpha}, \quad M_{\alpha\beta} = \varepsilon^{-1} \frac{\partial w}{\partial \kappa_{\alpha\beta}}, \quad P_\alpha = \varepsilon^{-2} \frac{\partial w}{\partial \psi_\alpha} \tag{45}$$

where $b = 5/6$. Since E_{33} has been determined from incompressibility, S_3 is a reactive quantity to be found from eqn (43b). Furthermore, note that, for the specific geometry considered in this paper, $\kappa_{12} = 0$ (eqns (9) and (10)), and so M_{12} is also a reactive quantity with its corresponding constitutive relation deleted from eqns (45). The same is true of $(\gamma_\alpha, g_\alpha)$ and (Q_α, S_α) if transverse shear deformation is neglected.

STRAIN–ENERGY DENSITY FUNCTION

The development of an approximate form of the two-dimensional strain–energy function w for an incompressible material follows the analyses of Simmonds (1986) and Taber (1987). After substitution of eqns (29), (33), and (36), eqns $(25)_1$–$(25)_8$ can be written in the form

$$E_n = \overset{0}{E}_n + \varepsilon \overset{1}{E}_n + \varepsilon^2 \overset{2}{E}_n + \cdots \tag{46}$$

where $n = 1, 2, \ldots, 8$ *and, subsequently, summation over double subscripts is implied over this range*. Again, the relation for E_{33} is not included due to the *a priori* enforcement of incompressibility. A Taylor series expansion provides

$$W(E_n) = \overset{0}{W} + \varepsilon \overset{0}{W}_1 + \tfrac{1}{2}\varepsilon^2 \overset{0}{W}_2 + \cdots \tag{47}$$

for the three-dimensional strain–energy density, where

$$W_n \equiv \frac{\partial^n W}{\partial \varepsilon^n}, \quad (^0) \equiv (\;)_{\varepsilon = 0}. \tag{48}$$

Simmonds (1985, 1986) has shown that, for moderately large bending and transverse shear strains, i.e. $(t\kappa_\alpha)^2$, γ_α^2, $g_\alpha^2 \ll 1$, the form of w for a thin shell is the same as that for a flat plate. Thus, the initial curvatures can be ignored, and the chain rule and eqn (46) yield

$$\overset{0}{W}_1 = \overset{0}{W}_{,n} \overset{1}{E}_n, \quad \overset{0}{W}_2 = \overset{0}{W}_{,mn} \overset{1}{E}_m \overset{1}{E}_n + 2 \overset{0}{W}_{,n} \overset{2}{E}_n \tag{49}$$

in which

† For an incompressible material, one can take $W = \bar{W}(E_{ij}) + p[(G/\bar{G})^{1/2} - 1]$, where p is a hydrostatic pressure and the term in brackets is given by eqns (35). Normally, p would be determined from the equilibrium equations and boundary conditions. But here, one assumes that $p \simeq \bar{\sigma}_{33}$, the *true* transverse normal stress, in agreement with the exact solution for bending of a cuboid (Green and Zerna, 1968). Based on this assumption, p contributes terms of higher order in the constitutive equations and, therefore, is dropped with incompressibility enforced *a priori* in W.

$$W_{,n} \equiv \frac{\partial W}{\partial E_n}. \tag{50}$$

Now, insertion of eqns (46) and (47) into eqn (44) and subsequent integration gives (with eqns (36) and (37))

$$\begin{aligned} w = 2\overset{0}{W}+\varepsilon^2\Big\{(\lambda_3^2/3)\Big[&\overset{0}{W}_{,11}\kappa_{11}^2+\overset{0}{W}_{,22}\kappa_{12}^2+\overset{0}{W}_{,44}\kappa_{21}^2+\overset{0}{W}_{,55}\kappa_{22}^2 \\ &+2\Big(\overset{0}{W}_{,12}\kappa_{11}\kappa_{12}+\overset{0}{W}_{,14}\kappa_{11}\kappa_{21}+\overset{0}{W}_{,15}\kappa_{11}\kappa_{22}+\overset{0}{W}_{,24}\kappa_{12}\kappa_{21} \\ &+\overset{0}{W}_{,25}\kappa_{12}\kappa_{22}+\overset{0}{W}_{,45}\kappa_{21}\kappa_{22}\Big)\Big]+b^{-1}\Big[\overset{0}{W}_{,33}\gamma_1^2+\overset{0}{W}_{,66}\gamma_2^2+\overset{0}{W}_{,77}g_1^2 \\ &+\overset{0}{W}_{,88}g_2^2+2\Big(\overset{0}{W}_{,36}\gamma_1\gamma_2+\overset{0}{W}_{,37}\gamma_1 g_1+\overset{0}{W}_{,38}\gamma_1 g_2+\overset{0}{W}_{,67}\gamma_2 g_1 \\ &+\overset{0}{W}_{,68}\gamma_2 g_2+\overset{0}{W}_{,78}g_1 g_2\Big)\Big]+F(\lambda_3')\Big\}+\cdots \end{aligned} \tag{51a}$$

where

$$F(\lambda_3') = (2\lambda_3\lambda_3'/3)\Big(\overset{0}{W}_{,16}\kappa_{11}+\overset{0}{W}_{,26}\kappa_{12}+\overset{0}{W}_{,46}\kappa_{21}+\overset{0}{W}_{,56}\kappa_{22}+\overset{0}{W}_{,66}\lambda_3'/\lambda_3\Big) \tag{51b}$$

and the $k_{\alpha\beta}$ have been replaced by $\kappa_{\alpha\beta}$ so that w vanishes for a shell in the undeformed state. Due to condition (27b), many terms have dropped out of the above equations. In fact, with the observation that shear strains must enter through terms of at least second order, eqns (25) and (49) show that

$$\int \overset{0}{W}_1 \, \mathrm{d}\bar{n} = 0.$$

Furthermore, if it is assumed that stresses depend on only the local values of the strains and not strain gradients, then $F(\lambda_3')$ can be dropped.

Equations (51) are valid for any W that allows axisymmetric deformation. One can now specialize eqn (51a) to a neo-Hookean shell with

$$W = \Lambda_{1L}^2+\Lambda_{2L}^2+\Lambda_{3L}^2-3 \tag{52}$$

in which Lagrange stretch ratios are defined by

$$\Lambda_{iL}^2 = \mathbf{G}_i\cdot\mathbf{G}_i/\mu_i^2 \quad (i \textit{ not summed}) \tag{53}$$

with $\mathbf{G}_i$ given by eqn (11a). For a flat plate, $\mu_i = 1$ and $\hat{E}_{ij} = E_{ij}$, and substitution of eqn (11a) into eqn (53) gives

$$\begin{aligned} \Lambda_{1L}^2 &= \Lambda_1^2+E_{12}^2+E_{13}^2 \\ \Lambda_{2L}^2 &= \Lambda_2^2+E_{21}^2+E_{23}^2 \\ \Lambda_{3L}^2 &= \Lambda_3^2+E_{31}^2+E_{32}^2 \end{aligned} \tag{54a}$$

in which

$$\Lambda_i = 1+E_{ii} \quad (i \textit{ not summed}). \tag{54b}$$

Now, substitution of eqns (54a) along with Λ_3, given by solving eqn (35b), into eqn (52) provides W as a function of $E_1, E_2, \ldots, E_8$. Then eqn (51a) becomes, upon restoration of dimensional variables

$$
\begin{aligned}
w/Ct = {} & \lambda_1^2+\lambda_2^2+\lambda_3^2+e_{12}^2+e_{21}^2-3+(\lambda_3^2t^2/12)[(1+3\lambda_2^2\lambda_3^4)\kappa_{11}^2+2(3\lambda_1\lambda_2\lambda_3-1)\lambda_3^3\kappa_{11}\kappa_{22} \\
& +(1+3\lambda_1^2\lambda_3^4)\kappa_{22}^2+(1+3e_{21}^2\lambda_3^4)\kappa_{12}^2+2(1+3e_{12}e_{21}\lambda_3)\lambda_3^3\kappa_{12}\kappa_{21}+(1+3e_{12}^2\lambda_3^4)\kappa_{21}^2 \\
& -6\lambda_3^4(\lambda_2e_{21}\kappa_{11}\kappa_{12}+\lambda_2e_{12}\kappa_{11}\kappa_{21}+\lambda_1e_{21}\kappa_{22}\kappa_{12}+\lambda_1e_{12}\kappa_{22}\kappa_{21})] \\
& +b^{-1}(\gamma_1^2+2\lambda_2\lambda_3^2\gamma_1g_1+g_1^2+\gamma_2^2+2\lambda_1\lambda_3^2\gamma_2g_2+g_2^2)
\end{aligned} \tag{55}
$$

where λ_3 is given by eqn (38a). For $\gamma_\alpha = g_\alpha = e_{12} = e_{21} = \kappa_{12} = \kappa_{21} = 0$, eqn (55) agrees with the result of Simmonds (1986).

At this point, we make a few observations. First, note that, if Q_α and S_α are computed through eqns (45) and (55), the shear correction factor b drops out. This is a consequence of taking parabolic distributions for the shear stresses *and* shear strains. Secondly, in a strictly two-dimensional formulation (Simmonds and Danielson, 1972; Reissner, 1974), the terms involving g_α in eqn (55) do not appear. For axisymmetric deformation, the shell theory of Reissner (1969a, b, 1972) and the work of Taber (1987) contain a transverse shear strain Γ defined as the rotation due to shear of lines originally normal to the reference surface. For small strains, a two-dimensional theory can be based on $\Gamma = \gamma_2+g_2$ and an equivalent transverse shear stress resultant $\bar{Q}_2 = Q_2+S_2$. In this case, as shown by eqn (55) with $\lambda_i \to 1$, w would contain a Γ^2 term. For large strain, however, the appearance of stretch ratios in the γ_2g_2 cross term apparently indicates that such a theory is not possible. When torsion is allowed, the γ_1g_1 and the complex κ_{12} and κ_{21} terms suggest similar conclusions. Moreover, one can note that the term γ_2^2/λ_2^2 in w as derived by Taber (1987) seems to represent something of a middle ground approximation for large strain. But experimental and numerical studies are needed to resolve the accuracy of the approximations proposed here.

SUMMARY

With $\kappa_{12} = 0$, eqns (9), (10), (26), (42), (43), and (45) provide essentially 33 equations to be solved for 33 unknowns: r, z, ϕ, Ω, ω, $e_{\alpha\beta}$, γ_α, g_α, $\kappa_{\alpha\beta}$ (except κ_{12}), ψ_α, $N_{\alpha\beta}$, Q_α, $M_{\alpha\beta}$, P_α, and S_i. The boundary conditions, which can be found through the principle of virtual work, are not considered here.

Equations (25) and (42) delineate the relative orders of various contributions to the shell behavior. The $O(1)$ terms provide the equations of non-linear membrane theory, the $O(\varepsilon)$ terms add bending and transverse shear effects, and the $O(\varepsilon^2)$ terms contain the higher-order effects due to moments turning about the "normal" (actually the $\mathbf{e}_3$ direction) to the reference surface. Since the ψ_α do not enter w to $O(\varepsilon^2)$ (eqns (51)), the latter terms probably can be neglected in most problems. Equations (42b) also suggest ignoring these terms, along with the q_i, especially since the effects of p_1 and p_2 are neglected. Finally, a theory neglecting transverse shear deformation can be recovered from these equations by stipulating $\gamma_\alpha = g_\alpha = 0$ in the strain–energy density function.

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REFERENCES

Green, A. E. and Zerna, W. (1968). *Theoretical Elasticity*, 2nd Edn, Chap. 3. Clarendon Press, Oxford.
Libai, A. and Simmonds, J. G. (1983). Nonlinear elastic shell theory. *Adv. Appl. Mech.* **23**, 271–371.
Naghdi, P. M. (1957). On the theory of thin elastic shells. *Q. Appl. Math.* **14**, 369–380.
Pietraszkiewicz, W. (1980). Finite rotations in the nonlinear theory of thin shells. In *Thin Shell Theory, New Trends and Applications* (Edited by W. Olszak), pp. 153–208. Springer, New York.
Reissner, E. (1950). On axisymmetrical deformations of thin shells of revolution. *Proc. Symp. Appl. Math.*, Vol. 3, pp. 27–52. American Mathematical Society.
Reissner, E. (1966). On the foundations of the theory of elastic shells. *Proc. 11th Int. Congr. Appl. Mech.*, pp. 20–30, 1964.
Reissner, E. (1969a). On the foundations of generalized linear shell theory. In *Theory of Thin Shells, Proc. IUTAM Symp.*, pp. 15–30. Copenhagen, 1967.
Reissner, E. (1969b). On finite symmetrical deflections of thin shells of revolution. *J. Appl. Mech.* **36**, 267–270.

Reissner, E. (1970). On the derivation of two-dimensional shell equations from three-dimensional elasticity theory. *Stud. Appl. Math.* **49,** 205–224.

Reissner, E. (1972). On finite symmetrical strain in thin shells of revolution. *J. Appl. Mech.* **39,** 1137–1138.

Reissner, E. (1974). Linear and nonlinear theory of shells. In *Thin Shell Structures* (Edited by Y. C. Fung and E. E. Sechler), pp. 29–44. Prentice-Hall, Englewood Cliffs, New Jersey.

Reissner, E. (1975). Note on the equations of finite-strain force and moment stress elasticity. *Stud. Appl. Math.* **54,** 1–8.

Schmidt, R. (1985). Mechanics of materials and structural theories. A current trend in shell theory: constrained geometrically nonlinear Kirchhoff–Love type theories based on polar decomposition of strains and rotations. *Comput. Struct.* **20,** 265–275.

Simmonds, J. G. (1985). The strain energy density of rubber-like shells. *Int. J. Solids Structures* **21,** 67–77.

Simmonds, J. G. (1986). The strain energy density of rubber-like shells of revolution undergoing torsionless, axisymmetric deformation (axishells). *J. Appl. Mech.* **53,** 593–596.

Simmonds, J. G. (1987). The strain energy density of compressible, rubber-like axishells. *J. Appl. Mech.* **54,** 453–454.

Simmonds, J. G. and Danielson, D. A. (1972). Nonlinear shell theory with finite rotation and stress-function vectors. *J. Appl. Mech.* **39,** 1085–1090.

Taber, L. A. (1987). Large elastic deformation of shear deformable shells of revolution: theory and analysis. *J. Appl. Mech.* **54,** 578–584.

APPENDIX: ALTERNATE FORM OF THE EQUILIBRIUM EQUATIONS

Reissner (1950) showed that the equations of force equilibrium for axisymmetric deformation of shells of revolution simplify considerably when written in terms of horizontal and vertical force components. Replacing eqn $(18)_1$, one can write

$$\begin{aligned} \mathbf{N}_s &= H_s\hat{\mathbf{e}}_r + V_s\hat{\mathbf{e}}_z + N_{s\theta}\hat{\mathbf{e}}_\theta \\ \mathbf{N}_\theta &= H_\theta\hat{\mathbf{e}}_r + V_\theta\hat{\mathbf{e}}_z + N_{\theta\theta}\hat{\mathbf{e}}_\theta \\ \mathbf{p} &= p_H\hat{\mathbf{e}}_r + p_V\hat{\mathbf{e}}_z + p_\theta\hat{\mathbf{e}}_\theta \end{aligned} \tag{A1}$$

where the $\hat{\mathbf{e}}_i$ are given by eqns (6a). Substitution into eqn $(20)_1$, taking appropriate derivatives of the unit vectors, and using eqns (9b) yield the scalar relations

$$\begin{aligned} (\bar{r}V_s)' + \bar{r}p_V &= 0 \\ (\bar{r}H_s)' - N_{\theta\theta} - \bar{r}\Omega' N_{s\theta} + \bar{r}p_H &= 0 \\ (\bar{r}N_{s\theta})' + H_\theta + \bar{r}\Omega' H_s + \bar{r}p_\theta &= 0 \end{aligned} \tag{A2}$$

in place of eqns (42a). Given p_V, the first relation can be integrated to obtain V_s directly.